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1990 J. Phys. A: Math. Gen. 23 L905

(http://iopscience.iop.org/0305-4470/23/17/011)

J. Phys. A: Math. Gen. 23 (1990) L905-L912. Printed in the UK

LETTER TO THE EDITOR

Exact solution of the $A_n^{(2)}$ lattice models

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Received 14 May 1990

Abstract. The exact solution of the $A_n^{(2)}$ integrable lattice models is presented. The free energy and excitation energies of the vertex models are found as functions of the spectral and anisotropy parameters as well as the finite-size corrections yielding the central charges and conformal dimensions.

These lattice models yield a new massive QFT where the mass spectrum and the S-matrix are obtained through the light-cone approach.

The ground state and excitation energies of a new anisotropic spin-one integrable chain associated to the $A_2^{(2)}$ model are computed.

The exact resolution of 2D integrable statistical models has made impressive progress in recent years [1]. The interest in such systems has been further enhanced by the discovery that their scaling behaviours (massless and massive) are universal.

In this letter we present the exact Bethe ansatz solution of the $A_n^{(2)}$ vertex models ($n \geqslant 2$). These are two-dimensional vertex models on a square lattice where each bond can be in $(n+1)$ different states ($n+1$ colours) [2, 3].

We compute the free energy and the excitation energies in closed form. Both scaling limits (massive and massless) are explicitly calculated. Using the light-cone approach [4], we find a relativistic QFT as a scaling limit of the $A_n^{(2)}$ vertex models. We explicitly derive the mass spectrum and S-matrix for these new field theories.

Finally we study the massless continuous limit of the $A_n^{(2)}$ vertex models using the methods of [5, 6, 7]. We find that the underlying conformal field theory is like an n-component Coulomb gas (central charge $= n/2$ for even n and $c=(n+1)/2$ for odd n, $n \geqslant 3$) where the conformal dimensions are given by (28).

The $A_n^{(2)}$ vertex models are trigonometric or hyperbolic solutions of the Yang–Baxter equation depending on two non-trivial parameters: θ (spectral parameter) and γ (anistropy or deformation parameter) [2]. The Boltzmann weights can be found in [2] where $q=\mathrm{e}^{\mathrm{i}\gamma}$, $x=\mathrm{e}^{2\mathrm{i}\theta}$ and $\xi=-\mathrm{e}^{\mathrm{i}\gamma(n+1)}$ for the trigonometric regime. The weights are manifestly real for all n in the hyperbolic regime and for $A_2^{(2)}$ in the trigonometric domains. The Bethe ansatz equations can be written for $n=2$ ($A_2^{(2)}$) as:

$$\left(\frac{\cosh(\lambda_k+\mathrm{i}\gamma/2)}{\cosh(\lambda_k-\mathrm{i}\gamma/2)}\right)^N=\prod_{j=1}^{m}\frac{\sinh(\lambda_k-\lambda_j+\mathrm{i}\gamma)\cosh(\lambda_k-\lambda_j-\mathrm{i}\gamma/2)}{\sinh(\lambda_k-\lambda_j-\mathrm{i}\gamma)\cosh(\lambda_k-\lambda_j+\mathrm{i}\gamma/2)}. \tag{1}$$

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0305-4470/90/170905+08$03.50

The BAE follows for any n from the zero-residue condition applied to the transfer matrix eigenvalues proposed in [3]. We find

$$\delta_{l,1}\left(\frac{\cosh(\lambda_k^{(1)}+\mathrm{i}\gamma/2)}{\cosh(\lambda_k^{(1)}-\mathrm{i}\gamma/2)}\right)^N + \sum_{\alpha=2}^{(n+\varepsilon)/2-1}\delta_{l,\alpha}$$
$$+\delta_{l,(n+\varepsilon)/2}\left[\prod_{j=1}^{m(n-\varepsilon)/2}\left((\varepsilon+1)\frac{\cosh(\lambda_k^{(n/2)}-\lambda_j^{(n/2)}+\mathrm{i}\gamma/2)}{\cosh(\lambda_k^{(n/2)}-\lambda_j^{(n/2)}-\mathrm{i}\gamma/2)}\right.\right.$$
$$\left.\left.-\varepsilon\frac{\cosh(\lambda_k^{(n-1)/2}-\lambda_j^{(n+1)/2}-\mathrm{i}\gamma/2)\cosh 2(\lambda_k^{(n-1)/2}-\lambda_j^{(n+1)/2}+\mathrm{i}\gamma/2)}{\cosh(\lambda_k^{(n-1)/2}-\lambda_j^{(n+1)/2}+\mathrm{i}\gamma/2)\cosh 2(\lambda_k^{(n-1)/2}-\lambda_j^{(n+1)/2}-\mathrm{i}\gamma/2)}\right)\right]$$
$$-\varepsilon\delta_{l,(n+1)/2}\left\{\prod_{j=1}^{m(n+1)/2}\frac{\cosh(\lambda_k^{(n+1)/2}-\lambda_j^{(n+1)/2}-\mathrm{i}\gamma)}{\cosh(\lambda_k^{(n+1)/2}-\lambda_j^{(n+1)/2}+\mathrm{i}\gamma)}\right.$$
$$\left.\times\prod_{j=1}^{m(n-1)/2}\frac{\cosh(\lambda_k^{(n+1)/2}-\lambda_j^{(n-1)/2}-\mathrm{i}\gamma/2)\cosh 2(\lambda_k^{(n+1)/2}-\lambda_j^{(n-1)/2}+\mathrm{i}\gamma/2}{\cosh(\lambda_k^{(n+1)/2}-\lambda_j^{(n-1)/2}+\mathrm{i}\gamma/2)\cosh 2(\lambda_k^{(n+1)/2}-\lambda_j^{(n-1)/2}-\mathrm{i}\gamma/2)}\right\}$$
$$=(-1)^{\varepsilon+1}\prod_{j=1}^{m_l}\frac{\sinh(\lambda_k^{(l)}-\lambda_j^{(l)}+\mathrm{i}\gamma)}{\sinh(\lambda_k^{(l)}-\lambda_j^{(l)}-\mathrm{i}\gamma)}$$
$$\times\prod_{\sigma=\pm1}\prod_{j=1}^{m_{l+\sigma}}\frac{\cosh(\lambda_k^{(l)}-\lambda_j^{(l+\sigma)}-\mathrm{i}\gamma/2)}{\cosh(\lambda_k^{(l)}-\lambda_j^{(l+\sigma)}+\mathrm{i}\gamma/2)} \tag{2}$$

where $1\leqslant k\leqslant m_l$, $1\leqslant l\leqslant n$, $\varepsilon=0$ if n is even and $\varepsilon=-1$ if n is odd. Here $n\geqslant 3$.

The eigenvalues of the transfer matrix expressed in terms of the roots $\{\lambda_k^{(l)}, 1\leqslant k\leqslant m_l, 1\leqslant l\leqslant n\}$ of the BAE (1), (2) as:

$$\Lambda(\theta,\lambda_k^{(l)})=\left\{\sin(\theta-\gamma)\cos\left[\theta-\left(\frac{n+1}{2}-\varepsilon\right)\gamma\right]\right\}^N A(\theta)$$
$$+\left\{\sin\theta\cos\left[\theta-\left(\frac{n+1}{2}-\varepsilon\right)\gamma\right]\right\}^N\sum_{i=1}^{n-1}B_i(\theta)$$
$$+\left\{\sin\theta\cos\left[\theta-\left(\frac{n-1}{2}\right)\gamma\right]\right\}^N C(\theta) \tag{3}$$

where

$$A(\theta)=\prod_{j=1}^{m_1}\frac{\sinh(\lambda_j^{(1)}+\mathrm{i}\theta+\mathrm{i}\gamma/2)}{\sinh(\lambda_j^{(1)}+\mathrm{i}\theta-\mathrm{i}\gamma/2)}\qquad 1\leqslant l\leqslant\frac{n-1}{2}+\frac{\varepsilon}{2}$$
$$B_l(\theta)=\prod_{j=1}^{m_l,m_{l+1}}\frac{\sinh[\lambda_j^{(l)}+\mathrm{i}\theta-\mathrm{i}((l/2)+1)\gamma]\sinh[\lambda_j^{(l+1)}+\mathrm{i}\theta-(l-1)\gamma/2]}{\sinh[\lambda_j^{(l)}+\mathrm{i}\theta-\mathrm{i}l\gamma/2]\sinh[\lambda_j^{(l+1)}+\mathrm{i}\theta-\mathrm{i}(l+1)\gamma/2]}$$
$$B_{n/2}(\theta)=\prod_{j=1}^{m(n/2)}\frac{\sinh[\lambda_j^{(n/2)}+\mathrm{i}\theta-\mathrm{i}(n+4)\gamma/4]\cosh[\lambda_j^{(n/2)}+\mathrm{i}\theta-\mathrm{i}(n-2)\gamma/4]}{\sinh[\lambda_j^{(n/2)}+\mathrm{i}\theta-\mathrm{i}n\gamma/4]\cosh[\lambda_j^{(n/2)}+\mathrm{i}\theta-\mathrm{i}(n+2)\gamma/4]}\qquad\varepsilon=0$$
$$B_{(n+1)/2}(\theta)=\prod_{j=1}^{m(n-1)/2,m(n+1)/2}\frac{\sinh[\lambda_j^{(n-1)/2}+\mathrm{i}\theta-\mathrm{i}(n+3)\gamma/4]\times\sinh 2[\lambda_j^{(n+1)/2}+\mathrm{i}\theta-\mathrm{i}(n-3)\gamma/4]}{\sinh[\lambda_j^{(n-1)/2}+\mathrm{i}\theta-\mathrm{i}(n-1)\gamma/4]\times\sinh 2[\lambda_j^{(n+1)/2}+\mathrm{i}\theta-\mathrm{i}(n+1)\gamma/4]}\qquad\varepsilon=-1$$
$$C(\theta)=\prod_{j=1}^{m_1}\frac{\cosh(\lambda_j^{(1)}+\mathrm{i}\theta+\mathrm{i}(n+1-\varepsilon)\gamma/2)}{\cosh(\lambda_j^{(1)}+\mathrm{i}\theta+\mathrm{i}n\gamma/2)}$$
$$\overline{B_i(\bar{u})}=B_{n+\varepsilon-i}\left(\left(\frac{n+1}{2}\right)^{\gamma}+\frac{\mathrm{i}\pi}{2}-u\right).$$

Equations (1) hold for the $A_n^{(2)}$ models in the fundamental representation and they have been written in the trigonometric regime. That is, when the vertex weights are trigonometric functions of θ and γ. An analogous set of equations hold in the hyperbolic regime with trigonometric functions instead of hyperbolic functions in (1).

In the $N=\infty$ limit the roots $\lambda_k^{(l)}$ are disposed as a continuous distribution for the antiferromagnetic ground state. By standard methods [1] equations (1) become a set of linear integral equations for the densities of roots

$$\rho^{(l)}(\lambda_k^{(l)}) \equiv \lim_{N\to\infty} \frac{1}{N(\lambda_{k+1}^{(l)}-\lambda_k^{(l)})}.$$

We find for the ground state

$$\sigma_v^{(l)}(\lambda) - \sum_{j=1}^{(n-\varepsilon)/2} \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} k_{lj}(\lambda-\mu)\sigma_v^j(\mu) = \frac{\delta_{l1}}{2\pi}\bar{\phi}'(\lambda, \gamma/2) \tag{4}$$

where

$$k_{ll'} = \delta_{ll'}\phi'(\lambda,\gamma) + (\delta_{l,l'+1}+\delta_{l,l'-1}+\delta_{n/2,l}\delta_{ll'})\bar{\phi}'(\lambda,\gamma/2) \qquad \varepsilon=0$$

$$\begin{aligned} k_{ll'} = {} & \delta_{ll'}\phi'(\lambda,\gamma) - \delta_{l,(n+1)/2}\delta_{l,l'}\bar{\phi}'(\lambda,\gamma) + (\delta_{l,(n-1)/2}\delta_{l,l'+1}+\delta_{l,(n+1)/2}\delta_{l,l'-1})\bar{\phi}'(2\lambda,\gamma) \\ & + (\delta_{l,l'+1}+\delta_{l,l'-1}-\delta_{l,(n-1)/2}\delta_{l,l'+1}-\delta_{l,(n+1)/2}\delta_{l,l'-1})\bar{\phi}'(\lambda,\gamma/2) \end{aligned} \tag{5}$$

and $\phi'(\lambda,\alpha)$ stands for $d\phi/d\lambda$

$$\phi(\lambda,\alpha) = i\log\frac{\sinh(\lambda+i\alpha)}{\sinh(\lambda-i\alpha)} \qquad \bar{\phi}(\lambda,\alpha) = i\log\frac{\cosh(\lambda-i\alpha)}{\cosh(\lambda+i\alpha)}.$$

In the presence of a hole, say at $\lambda=\theta_h$ in the level l', the densities of real roots become:

$$\rho^{(l)}(\lambda) = \sigma_v^{(l)}(\lambda) + \frac{1}{N}[-\delta(\lambda-\theta_n)\delta_{ll'} + \sigma_{ll'}(\lambda)] \tag{6}$$

where $\sigma_v(\lambda)$ stands for the density in the ground state (vacuum) and $\sigma_{ll'}(\lambda)$ fulfils the system of equations $0\leqslant\gamma\leqslant\pi/2$.

$$\sigma_{ll'}(\lambda) - \sum_{j=1}^{(n-\varepsilon)/2} \int_{-\infty}^{+\infty} d\mu\, k_{lj}(\lambda-\mu)\sigma_{jl'}(\mu) = k_{ll'}(\lambda-\theta_n). \tag{7}$$

We will not consider here the possibility of complex roots and we will limit ourselves to the interval.

Solving equations (3) and (7) by Fourier transform we find:

$$\begin{aligned} \tilde{\sigma}_v^{(l)}(k) &= \frac{\cosh(k/4)[(n+1-2l)(\pi-\gamma)+\varepsilon\pi]}{\cosh(k/4)[(n+1)(\pi-\gamma)+\varepsilon\pi]} \qquad 1\leqslant l\leqslant\frac{n+\varepsilon}{2} \\ \tilde{\sigma}_v^{(n+1)/2}(k) &= \frac{1}{2\cosh(k/4)[(n-1)(\pi-\gamma)-\pi]} \qquad \varepsilon=-1. \end{aligned} \tag{8}$$

That is,

$$\begin{aligned} \sigma_v^{(l)} = {} & \frac{4}{(n+1)(\pi-\gamma)+\varepsilon\pi}\sin\frac{l(\pi-\gamma)}{[(n+1)(\pi-\gamma)+\varepsilon\pi]}\cosh\frac{\lambda\pi}{2[(n+1)(\pi-\gamma)+\varepsilon\pi]} \\ & \times\left(\cosh\frac{\lambda\pi}{[(n+1)(\pi-\gamma)+\varepsilon\pi]} - \cos\frac{2l(\pi-\gamma)}{[(n+1)(\pi-\gamma)\to\varepsilon\pi]}\right)^{-1}. \end{aligned} \tag{9}$$

Notice that $m_l = N$ $(1\leqslant l\leqslant n)$ in the ground state.

The hole contribution to the roots density $\sigma_{ll}(\lambda)$ (see (6)) turns out to be

$$\tilde{\sigma}_{ll'}(k)=\delta_{ll'}-\frac{\sinh k\pi/2}{\sinh k\gamma/2}\sinh(l<x)\cosh\left[\left(\frac{n+1}{2}-l+\frac{\varepsilon\pi}{2(\pi-\gamma)}\right)x\right]$$
$$\times\left\{\sinh x\cosh\left[\left(\frac{n+1}{2}+\frac{\varepsilon\pi}{2(\pi-\gamma)}\right)x\right]\right\}^{-1}\qquad 1\leqslant l,l'\leqslant\frac{n+\varepsilon}{2}\tag{10}$$

$$\tilde{\sigma}_{nl}(k)=\frac{\tilde{\sigma}_{n-1,l}(k)}{2\cosh k(\pi/2-\gamma)/2}\qquad \varepsilon=-1$$

where $x\equiv k(\pi-\gamma)/2$.

We find the transfer matrix eigenvalues using densities (8) and (10). The free energy per site is

$$f_n(\theta,\gamma)=\int_0^\infty\frac{d\omega}{\omega}\sinh 2\omega\theta\frac{\cosh[((n-1)(\pi-\gamma)/2+\varepsilon\pi/2)\omega]}{\cosh[((n+1)(\pi-\gamma)/2+\varepsilon\pi/2)\omega]}\frac{\sinh\gamma\omega}{\sinh\pi\omega}.\tag{11}$$

More generally, the excited eigenvalues have the following form for $N\to\infty$

$$\Lambda(\theta,\gamma)\underset{N\to\infty}{=}\exp[-Nf_n(\theta,\gamma)-\mathrm{i}g_l(\theta,\theta_h,\gamma)]\tag{12}$$

where $g_{l'}(\theta,\theta_h,\gamma)$ denotes the contribution of a hole at θ_h in the level l', which proves to be:

$$g_l(\theta,\theta_h,\gamma)=2\tan^{-1}\left[\left(\sinh\frac{(\theta_h+\mathrm{i}\theta)\pi}{2[(n+1)(\pi-\gamma)+\varepsilon\pi]}\right)\left(\sin\frac{l(\pi-\gamma)}{[(n+1)(\pi+\gamma)+\varepsilon\pi]}\right)^{-1}\right].\tag{13}$$

We see from (13) that these models are gapless since $g_l(\theta,-\infty,\gamma)=0$. Therefore, one can construct a massive field theory from them using the light-cone approach [1, 4]. In this framework the energy and momentum eigenvalues are given by

$$\varepsilon\pm p=\lim_{\substack{a\to0\\ \pm\mathrm{i}\theta\to\infty}}\frac{g_l(\pm\theta,\theta_h,\gamma)}{a}\tag{14}$$

where a is the lattice spacing. Combining (13) and (14) yields

$$\varepsilon=\mu_l\cosh\varphi\qquad p=\mu_l\sinh\varphi\tag{15}$$

where

$$\mu_l=\mu\sin\frac{l(\pi-\gamma)}{[(n+1)(\pi-\gamma)+\varepsilon\pi]}\qquad 1\leqslant l\leqslant\frac{n+\varepsilon}{2}\tag{16}$$

$$\mu_{(n+1)/2}=\frac{\mu}{2}\qquad \varepsilon=-1$$

$$\mu=\lim_{a\to0}\exp\left(\frac{-\mathrm{i}\theta\pi}{2[(n+1)(\pi-\gamma)+\varepsilon\pi]}\right)\qquad \varphi=\theta_h\pi/2[(n+1)(\pi-\gamma)+\varepsilon\pi].\tag{17}$$

That is, ϕ is the physical rapidity and μ_l the physical masses of the particles. The limit $a\to0$, $\mathrm{i}\theta\to\infty$ is taken, as is usual in the light-cone approach, such that μ is a finite-mass scale.

The mass spectrum (16) is characteristic of the $A_n^{(2)}$ algebras. The $A_{2n}^{(2)}$ exhibits the same mass spectrum as the Toda QFT whereas our $A_{2n-1}^{(2)}$ spectrum gives the one of the Toda QFT only $\gamma\to 0$ [8].

The S-matrix between a hole at branch l and another one at l' follow from (10) applying the method of [9]. (That is the S-matrix between a particle m_l and a particle $m_{l'}$.) It reads $S_{ll'}(\phi)=\exp[\mathrm{i}\delta_{ll'}(\phi)]$ where ϕ is the relativistic rapidity and

$$\delta_{ll'}(\varphi)=2\pi\int_0^{\varphi[2[(n+1)(\pi-\gamma)+\varepsilon\pi]/\pi]}\sigma_{ll'}(\lambda)\,\mathrm{d}\lambda. \tag{18}$$

Looking at (10) and (18) shows that this S-matrix is not an elementary function of ϕ whereas only hyperbolic functions appear in the Toda field theories for simply laced algebras [8, 10].

Besides this scaling limit yielding an integrable massive QFT, we can take the trivial continuous limit $(a\to 0)$ leading to the conformal invariant model.

Let us sketch the derivation of the finite-size corrections and give the results. The finite-size corrections to the free energy can be expressed in a form analogous to the $A_n^{(1)}$ model [7].

$$\begin{aligned}L_N(\theta)&=f_N(\theta)-f_\infty(\theta)\\&=-\sum_{l=1}^{(n-\varepsilon)/2}\left(\int_{-\infty}^{-\Lambda_l^-}+\int_{\Lambda_l^+}^{+\infty}\right)\mathrm{d}\lambda_l\,f_l(\lambda_l)\sigma_N^{(l)}(\lambda_l)\\&\quad+\frac{1}{2N}\sum_{l=1}^{(n-\varepsilon)/2}[f_l(\Lambda_l^+)+f_l(\Lambda_l^-)]\\&\quad+\frac{1}{12N^2}\sum_{l=1}^{(n-\varepsilon)/2}\left(\frac{f'(\Lambda_l^+)}{\sigma_N^{(l)}(\Lambda_l^+)}-\frac{f'(-\Lambda_l^-)}{\sigma_N^{(l)}(-\Lambda_l^-)}\right)\\&\quad+\text{higher orders}\end{aligned} \tag{19}$$

where $\pm\Lambda_l^{\pm}$ are the largest positive and negative roots of the BAE (1) in the lth branch

$$\sigma_N^{(l)}(\lambda)=\frac{\mathrm{d}Z_N^{(l)}}{\mathrm{d}\lambda}(\lambda) \tag{20}$$

and

$$\begin{aligned}Z_N^{(l)}(\lambda)=\frac{1}{2\pi N}\Bigg\{&\sum_{\sigma=\pm1}\sum_{j=1}^{\rho_{l+\sigma}}\bar{\phi}(\lambda-\lambda_j^{(l+\sigma)},\gamma/2)+\sum_{j=1}^{\rho_l}\phi(\lambda-\lambda_j^{(l)},\gamma)\\&+(\varepsilon+1)\delta_{l,n/2}\sum_{j=1}^{m(n/2)}\bar{\phi}(\lambda-\lambda_j^{(n/2)},\gamma/2)\\&-\varepsilon\Bigg[\delta_{l,(n-1)/2}\left(\sum_{j=1}^{m(n+1)/2}[\bar{\phi}(2(\lambda-\lambda_j^{(n+1)/2}),\gamma)-\bar{\phi}(\lambda-\lambda_j^{(n+1)/2},\gamma/2)]\right)\\&-\delta_{l,(n+1)/2}\left(\sum_{j=1}^{m(n+1)/2}\bar{\phi}(\lambda-\lambda_{j_1}^{(n+1)/2},\gamma)-\sum_{j=1}^{m(n-1)/2}[\bar{\phi}(2(\lambda-\lambda_j^{(n-1)/2}),\gamma)\right.\\&\left.-\bar{\phi}(\lambda-\lambda_1^{(n-1)/2},\gamma/2)]\right)\Bigg]\Bigg\}.\end{aligned} \tag{21}$$

We define the Fourier transforms

$$x_l^{\mp}(\omega)=\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t\,\theta(\pm t)\sigma_N^{(l)}(\Lambda_l^{\mp}+t) \tag{22}$$

which are analytic functions of ω for $\pm\mathrm{Im}\,\omega > 0$. We get a matrix Riemann–Hilbert problem from the BAE (1), (2) approximated as (19)

$$x_k^-(\omega)+\sum_{l=1}^{(n-\varepsilon)/2}\hat{R}_{kl}(\omega)x_l^+(\omega)\,\mathrm{e}^{\mathrm{i}\omega(\Lambda_l^+-\Lambda_k^+)}$$
$$=\mathrm{e}^{-\mathrm{i}\omega\Lambda_k^+}\hat{\sigma}_k(\omega)+\frac{1}{2N}\left(-1+\sum_{l=1}^{(n-\varepsilon)/2}\hat{R}_{kl}(\omega)\,\mathrm{e}^{\mathrm{i}\omega(\Lambda_l^+-\Lambda_k^+)}\right)$$
$$\times\frac{-\mathrm{i}\omega}{12N^2}\sum_{l=1}^{(n-\varepsilon)/2}(\delta_{kl}-\mathrm{e}^{\mathrm{i}\omega(\Lambda_l^+-\Lambda_k^+)})\hat{R}_{kl}(\omega)(\sigma_N^{(l)}(\Lambda_l^+))^{-1} \tag{23}$$

where $R_{ll'}(\omega)=\delta_{ll'}-\sigma_{ll'}(\omega)$. The resolution of this problem is analogous to [7]. The new thing is that for the twisted case considered here, it is convenient to introduce a quantity

$$m_{(n-\varepsilon)/2+1}=\begin{cases}m_{n/2} & \text{even } n\\ m_{(n-1)/2}-2m_{(n+1)/2} & \text{odd } n.\end{cases} \tag{24}$$

In this way the weights of a Bethe ansatz state can be written as

$$S_k=\sum_{l=1}^{(n-\varepsilon)/2+1}M_{kl}m_l \qquad 1\leqslant k\leqslant\frac{n-\varepsilon}{2}+1 \tag{25}$$

where M stands for the extended Cartan matrix

$$M_{kl}=2\delta_{kl}-\delta_{k,l+1}-\delta_{k,l-1}+\varepsilon\delta_{k,(n-1)/2}\delta_{k,l+1}. \tag{26}$$

Notice that this matrix appears in the $R^{-1}(\omega)$ matrix

$$R^{-1}(\omega)=\frac{\sinh\omega\gamma/2}{\sinh\omega\pi/2}\left(\frac{M-2\cosh\omega(\pi-\gamma)}{2}\right). \tag{27}$$

Let us now obtain the finite-size corrections $L_N(\theta)$ for an excited state with weights S_l $(1\leqslant l\leqslant n)$ and $h_\pm^l$ holes beyond $\pm\Lambda_l^\mp$. That is a generic low-energy excitation (cf (13)).

Proceeding as in [7] we find

$$L_N(\theta)=-\frac{\pi}{N^2}\frac{(n-\varepsilon)}{12}\sin(K\theta)-\frac{2\pi\mathrm{i}}{N^2}[\Delta\,\mathrm{e}^{-\mathrm{i}K\theta}-\bar{\Delta}\,\mathrm{e}^{\mathrm{i}K\theta}] \tag{28}$$

where

$$K=\frac{\pi}{2[(n+1)(\pi-\gamma)+\varepsilon\pi]}$$
$$\Delta=\frac{2\pi}{\gamma}\sum_{l,l'=1}^{(n-\varepsilon)/2}\left(h_+^l-\frac{\gamma}{\pi}S_l\right)(M^{-1})_{ll'}\left(h_+^{l'}-\frac{\gamma}{\pi}S_{l'}\right) \tag{29}$$

$\bar{\Delta}$ follows from Δ by exchanging $h_+^l\leftrightarrow h_-^l$. Since the speed of sound here is $v=\sin(K\theta)$ (see (13)), (28) tells us that the central charge is $c=(n-\varepsilon)/2$. We recall that the parameter K gives the finite renormalization of the rapidity (cf (17)). It is usually connected with the one-loop β function in the associated QFT [4].

The results for c, Δ and $\bar{\Delta}$ indicate that the conformal behaviour is like a $(n-\varepsilon)/2$ component Coulomb gas. For the $A_2^{(2)}$ model we find $c=1$ and the usual Coulomb gas.

An integrable magnetic chain follows from the $A_{2n}^{(2)}$ vertex model as usual from the logarithmic derivative of the transfer matrix at $\theta = 0$. Therefore, the eigenvalues of this family of Hamiltonians can be trivially obtained from our results as

$$E = -\sin\gamma \frac{\partial}{\partial\theta} \log \Lambda(\theta)|_{\theta=0} \tag{30}$$

where $\Lambda(\theta)$ is given by (3), (11), (13), (19) and (28).

Let us briefly discuss the $n=2$ case where the 'spin' at each site has three components. Therefore we can write the Hamiltonian in terms of spin-one operators (S^x, S^y, S^z) as follows

$$\begin{aligned} H_1 = -\sum_{j=1}^{N} \Bigg\{ \sigma_j + f(\gamma) \Bigg[& (\cos\gamma - 1)\sigma_j^z - 2(1 + 2\cos\gamma)(S_j^z)^2 \\ & + \sigma_j^z(S_j^z - S_{j+1}^z) + \left(2(1+\cos\gamma/2) + \frac{1}{\cos\gamma - 1}\right)(\sigma_j)^2 \\ & - 2\left(\cos\frac{\gamma}{2} + 1\right)(\sigma_j^\perp)^2 + \left(3\cos\gamma - 2\cos\frac{\gamma}{2} - 1\right)(\sigma_j^z)^2 \Bigg] \Bigg\} \end{aligned} \tag{31}$$

where

$$\begin{aligned} & f(\gamma) = \frac{\cos\gamma - 1}{2\cos\gamma - 1} \\ & \sigma_j^z \equiv S_{j+1}^z S_j^z \qquad \sigma_j \equiv \sum_{a=x,y,z} S_{j+1}^a S_j^a \qquad \sigma_j^\perp \equiv \sigma_j - \sigma_j^z. \end{aligned} \tag{32}$$

This is a new spin-one integrable Hamiltonian. The ground-state energy follows from (11) and (30) for $n=2$

$$E(\gamma) = \frac{4\sin\gamma}{\gamma} \int_0^\infty dx \frac{\sinh x}{\sinh \pi x/\gamma} \frac{1}{2\cosh x/\gamma(\pi - \gamma) - 1}. \tag{33}$$

The excitation energies follow from (13) and (30) for $n=2$

$$\begin{aligned} & \varepsilon(\varphi, \gamma) = \frac{4\pi}{\sqrt{3}\,(\pi - \gamma)} \frac{\sinh(4\pi\varphi/3(\pi-\gamma))}{\sinh(2\pi\varphi/(\pi-\gamma))} \\ & \rho(\varphi, \gamma) = 2\tan^{-1}\left(\sinh\frac{2\pi\varphi}{3(\pi-\gamma)}\right) \end{aligned} \tag{34}$$

which yields the dispersion relation:

$$\varepsilon(\rho) = \frac{8\pi}{\sqrt{3}(\pi-\gamma)} \frac{\cos(\rho/2)}{4 - \cos^2(\rho/2)}. \tag{35}$$

One of us (EL) is supported in part by CAPES (Brazil).

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